

Complemented basic sequences in Frechet spaces with finite dimensional decomposition

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Abstract: Let E be a Frechet-Montel space and $(E_n)_{n \in \mathbb{N}}$ be a finite dimensional unconditional decomposition of E with $\dim(E_n) \leq k$ for some fixed $k \in \mathbb{N}$ and for all $n \in \mathbb{N}$. Consider a sequence $(x_n)_{n \in \mathbb{N}}$ formed by taking an element x_n from each E_n for all $n \in \mathbb{N}$. Then $(x_n)_{n \in \mathbb{N}}$ has a subsequence which is complemented in E .

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1 Introduction

Dubinsky has professed that approximation of an infinite dimensional space by its finite dimensional subspaces is one of the basic tenets of Grothendieck's mathematical philosophy in creating nuclear Frechet spaces [4]. That approach encompasses bases, finite dimensional decompositions and the bounded approximation property of spaces and raises questions about conditions for which a space must satisfy in order to admit such structures and also about the relation of existence or absence of these structures with subspaces, quotient spaces, complemented subspaces, etc. of the space.

He has ended his survey with a list of problems that were settled and open at that time [4], that is, in 1989. For example, it has been shown that quotient

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spaces of a nuclear Frechet space may not have a basis [6], or similarly, that a nuclear Frechet space does not necessarily have a basis [1]. On the question of complemented subspace with basis, the following was the most general result known at that time [4]. A nuclear Frechet space with a decomposition into two dimensional blocks contains a complemented basic sequence [5], and a shorter proof of the same result by utilizing Ramsey Theorem was obtained by Ketonen and Nyberg in [8], which also includes a remark to the effect that the method of proof cannot be extended to spaces with decompositions into higher dimensional blocks. On the other hand, Krone and Walldorf showed that complemented subspaces with a strong finite-dimensional decomposition of nuclear Köthe spaces have a basis [9].

Our first aim in this manuscript is to show that nuclear Frechet spaces which have decompositions of k dimensional blocks contain a complemented basic sequence, where $k \in \mathbb{N}$. In fact, we have proved the following : Let E be a Frechet-Montel space and $(E_n)_{n \in \mathbb{N}}$ be a finite dimensional decomposition of E with $\dim(E_n) \leq k$ for some fixed $k \in \mathbb{N}$ and for all $n \in \mathbb{N}$. Consider a sequence $(x_n)_{n \in \mathbb{N}}$ formed by taking an element x_n from each E_n for all $n \in \mathbb{N}$. Then $(x_n)_{n \in \mathbb{N}}$ has a subsequence which is complemented in E .

After preliminaries in section 2, we establish the above result in section 3.

2 Preliminaries

Recall that a locally convex (topological vector) space (over the field \mathbb{R} or \mathbb{C}) E is called a *Frechet space* if it is a complete and metrizable.

If E is metrizable, then E has a countable basis of neighbourhoods of 0 and if E is also locally convex, we may choose a basis with elements that are absolutely convex, say $\mathcal{B} = \{U_n : n \in \mathbb{N}\}$ with $U_1 \supseteq U_2 \supseteq \dots$. By setting $p_n(\cdot) := \|\cdot\|_{U_n}$ we obtain an increasing sequence $(p_n)_{n \in \mathbb{N}}$ of continuous seminorms. Thus, the topology of E can be given by an increasing sequence of continuous norms, a *fundamental system of seminorms*.

A locally convex space whose topology is defined using a countable set of compatible norms i.e. norms such that if a sequence that is fundamental in the norms and converges to zero in one of these norms, then it also converges to zero in the other, is called *countably normed*. Note that for a countably normed space E , $(\hat{E}, \|\cdot\|_{k+1}) \rightarrow (\hat{E}, \|\cdot\|_k)$, where \hat{E} denotes closure of

E , is an injection and all spaces we deal in this manuscript are countably normed.

A subspace Y of a topological space X is called *complemented* if Y is image of a continuous projection.

A sequence (x_n) in a Frechet space E is called a *basic sequence* if it is a Schauder basis of its closed linear span $\overline{\text{span}\{x_n : n \in \mathbb{N}\}}$, which we also denote by $[x_n]_{n \in \mathbb{N}}$.

A *finite dimensional decomposition* (FDD in short) in E is a sequence $(A_n)_{n \in \mathbb{N}}$ of continuous linear operators $A_n : E \rightarrow E$ such that $\dim(A_n(E)) < \infty$, $A_n A_m = \delta_{mn} A_n$ and $x = \sum_{n=1}^{\infty} A_n x$ for all $x \in E$.

A k -FDD is an FDD in which $(A_n)_{n \in \mathbb{N}}$ can be chosen such that $\dim(A_n(E)) \leq k$ for some $k \in \mathbb{N}$. A *strong FDD* is an FDD which is a k -FDD for some $k \in \mathbb{N}$.

Suppose that E has strong FDD property.

Then we define $E_n := \overline{A_n(E)}$, for all $n \in \mathbb{N}$ and let M be an infinite subset of \mathbb{N} . Denote $E_M = \bigoplus_{n \in M} E_n$ so that $E = E_{\mathbb{N}}$

Let G be a subspace of E . If $G = \overline{\bigoplus_{n \in \mathbb{N}} (G \cap E_n)}$, then we call G a *step subspace* of E .

We denote a sequence $(x_n)_{n \in \mathbb{N}}$ formed by taking an element x_n from each E_n for all $n \in \mathbb{N}$ by $(x_{E_n})_{n \in \mathbb{N}}$. We define a quotient space of E obtained by 'dividing' E by a step subspace as *step quotient* of E .

A complemented step subspace K of a Frechet space E is called *naturally complemented* if complement of K can be made a step subspace. Equivalently, let $P : E \rightarrow E$ be a projection with $P(E) = K$. Then K is naturally complemented if P commutes with any projection map A_n defined above on E .

ω denotes the space of all sequences with real elements. We refer to [11] for any unexplained definitions and notations.

3 Main result

The following lemmata are needed, and since the first one is well known, it is given without proof.

Lemma 1. *Let E, F be topological vector spaces and $T : E \rightarrow F$ be a continuous linear operator. Suppose restriction of T to a subspace M of E is*

an isomorphism and $T(M)$ is complemented in F . Then M is complemented in E .

START

Although the following result, the existence of basic subsequence of certain sequences in a Banach spaces, is also known (see, for example [3]), we include here with a sketch of a proof.

Lemma 2. *Let (x_n) be a normalized injective sequence ($x_n \neq x_m$ for $n \neq m$) in a Banach space $(E, \|\cdot\|)$ such that $\lim_{n \rightarrow \infty} u(x_n) = 0$ for all $u \in B$ where B is a w^* dense, i.e. $\sigma(E, E')$ dense, subset of E' . Then, there exists an infinite subset M of \mathbb{N} such that $(x_n)_{n \in M}$ is a basic sequence in E .*

Proof. Since $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $x_n \rightarrow 0$ in $\sigma(E, B)$, (x_n) does not have any $\|\cdot\|$ -Cauchy subsequence. Otherwise, if a subsequence converges to x for some $x \in E$, then $u(x) = 0$ for $u \in B$ implies that $\|x\| = 0$ contradicting the fact that $\|\cdot\|$ is continuous.

Thus, there exists an infinite subset M of \mathbb{N} such that either $(x_n)_{n \in M}$ is equivalent to a basic sequence in l_1 or $(x_n)_{n \in M}$ is weakly Cauchy with respect to $\sigma(E, E')$ by Rosenthal's l_1 theorem. If former is the case, then (x_n) has a basic subsequence.

For the latter case, we may assume, WLOG, (x_n) itself is a weakly Cauchy sequence. Now, suppose the bounded Cauchy sequence (x_n) converge to $x \in E''$ in $\sigma(E'', E)$. If $x \in E$, then $x = 0$ by the assumption that $x_n \rightarrow 0$ in $\sigma(E, B)$. Thus, (x_n) is a weakly null normalized sequence and by Bessaga-Pelczynski selection principle [2], we can obtain a basic subsequence of (x_n) . If $x \in E'' \setminus E$, then the sequence $(x - x_n) \rightarrow 0$ in $\sigma(E'', E')$ (see [3] Chapter IV, or Ex. 10 in page 55). Thus, we may obtain a basic subsequence of $(x - x_n)$. WLOG, let us assume that $(x - x_n)$ is basic. Recall that $x \notin [x_n]$ (here in this lemma, $[\cdot]$ denotes the norm-closure of the span of \cdot). Also, we claim that we can find $k \in \mathbb{N}$ such that $x \notin [x - x_n]_{n \geq k}$. (If $x \in [x - x_n]_{n \geq 0}$ or $x \in [x - x_n]_{n \geq 1}$, we set $x = \sum_{n \in \mathbb{N}} a_n(x - x_n)$ and if a_{n_0} is the first nonzero term, we let $k = n_0 + 1$.) Now let P_1 and P_2 be continuous projections on $[x - x_n]_{n \geq k}$ and $[x_n]_{n \geq k}$ with kernels (containing hg?) $[x]$ and κ be a (the) basic constant (?) of $[x - x_n]_{n \geq k}$. We may assume $k = 1$ by reindexing. Let $m \in \mathbb{N}$ such that $\|\sum_{n=1}^m a_n x_n\| \leq 1$. Then, $\|\sum_{n=1}^m a_n(x_n - x) + \sum_{n=1}^m a_n x\| \leq 1$. But $\|\sum_{n=1}^m a_n(x_n - x)\| \leq \|P_1\|$, the operator norm of P_1 . Thus, there exists

$l \in \mathbb{N}$ such that $\| \sum_{n=1}^l a_n(x_n - x) \| \leq \| P_1 \| \cdot \kappa$ (?). Hence, $\| \sum_{n=1}^l a_n x_n - \sum_{n=1}^l x \| \leq \| P_1 \| \cdot \kappa$. Therefore, $\| \sum_{n=1}^l a_n x_n \| \leq \| P_1 \| \cdot \kappa + \| P_2 \|$ which shows that $(x_n)_{n \in \mathbb{N}}$ is a basic sequence with basic constant $\| P_1 \| \cdot \kappa + \| P_2 \|$ (?). Since $x_n \in E$, $[x_n] \subseteq E$, which is to say, x_n is a basic sequence in E . \square

In the above proof, we refer to Rosenthal l_1 - theorem, but actually we do not need to do that. Since $x_n \succ (?)x$ in $\sigma(E'', E')$, $y \in \bigcap_{k \geq n} [x_m]_{m \geq k}$ with respect to $\sigma(E'', E')$. Hence, $[x_n - x] \succ 0$ with respect to $\sigma(E'', E')$. Thus a similar approach as we claimed $\sigma(E, B)$ -converging to 0 by

We prefer that approach since in a countably normed Frechet space E , a subset B with the above property (ies) can easily be found.
(END)

It has been shown that the restriction of unbounded linear maps to infinite dimensional subspaces between some classes of Frechet spaces is an isomorphism. [10]

We shall show that for any sequence (x_n) in a Frechet space E and any linear map T , $(T(x_n))$ either has a subsequence, when normalized, is bounded or restriction of T on $[x_n]_{n \in \mathbb{N}}$ is an isomorphism.

Proposition 1. *Let E and F be Frechet spaces, $T : E \rightarrow F$ be a linear map and (x_n) be a sequence in E . Suppose E has a continuous norm and F is countably normed. (spaces with basis, their subspaces etc. are c. n.) Then there exists an infinite subset M of \mathbb{N} such that*

(i) $(\frac{1}{\|x_n\|} T x_n)_{n \in M}$ is a bounded subset of F for some continuous norm $\| \cdot \|$ on E ,

or

(ii) restriction of T to $[x_n]_{n \in M}$, that is, $T|_{[x_n]_{n \in M}}$ is an isomorphism and furthermore $[x_n]_{n \in M} = \lambda(A)$ for some nuclear Köthe space $\lambda(A)$.

Proof. Let $(\| \cdot \|_n)$ and $(| \cdot |_n)$ be increasing sequences of norms which define topologies of E and F , respectively, with $| T x |_k \leq \| x \|_k$ and F is countably normed. Without loss of generality, we may assume that $T|_{[x_n]_{n \in \mathbb{N}}}$ is an injection.

Set $M_0 = \mathbb{N}$. Consider the sequence $(\frac{T x_n}{\|x_n\|_1})_{n \in M}$. If it is bounded, we may take M to be \mathbb{N} and $k = 1$ to claim (i) holds.

Otherwise, $(\frac{Tx_n}{\|x_n\|_1})_{n \in M_0}$ is unbounded

Since $(\hat{F}, |\cdot|_{k+1}) \rightarrow (\hat{F}, |\cdot|_k)$ is an injection for all $k \in \mathbb{N}$ there exists $l_1 \in \mathbb{N}$ such that $\sup(\frac{|Tx_n|_{l_1}}{\|x_n\|_1}) = \infty$.

Thus, we can find an infinite subset M_1 of M such that $\sum_{n \in M_1} \frac{\|x_n\|_1}{|Tx_n|_{l_1}} < \infty$. Since $|Tx_n|_1 \leq \|x_n\|_1$, the sequence $(\frac{Tx_n}{|Tx_n|_{l_1}})_{n \in M_1}$ converges to zero. Thus, the normalized sequence $\frac{Tx_n}{|Tx_n|_{l_1}}$ converges to zero with respect to the topology $\sigma(F, F'_1)$ where F'_1 denotes the dual of the normed space $(F, |\cdot|_1)$. But F'_1 is $\sigma(F'_1, (\hat{F}_{l_1}, |\cdot|_{l_1}))$ dense. By applying Lemma 2, that is, considering E to be $(\hat{F}_{l_1}, |\cdot|_{l_1})$ and the sequence to be $(\frac{Tx_n}{|Tx_n|_{l_1}})_{n \in M_1}$ following the notation of Lemma 2, we obtain a basic subsequence of $(x_n)_{n \in M}$.

Now consider the sequence $(\frac{Tx_n}{\|x_n\|_{l_1}})_{n \in M_1}$. If it is bounded, by taking M to be M_1 and $k = l_1$ we may claim (i) holds.

Otherwise, there exists $l_2 \in \mathbb{N}$ such that $\lim_{n \rightarrow \infty} \frac{|Tx_n|_{l_2}}{\|x_n\|_{l_1}} = \infty$. Similar to above, we may find an infinite subset M_2 of M_1 such that the sequence $(\frac{Tx_n}{|Tx_n|_{l_2}})_{n \in M_2}$ converges to zero and we may use Bessaga-Pelczynski selection principle [2] to obtain an basic sequence in $(F, \|\cdot\|_{l_2})$ if necessary. Note that we may choose $|Tx_n|_k \leq \|x_n\|_k$ for all $k \in \mathbb{N}$, so that $l_2 > l_1$.

If we proceed in this manner, use Bessaga-Pelczynski selection principle [2] to obtain an basic sequence in $(F, \|\cdot\|_{l_k})$, if necessary, and assuming (i) does not hold in each step, we obtain an infinite decreasing sequence of (M_t) of infinite subsets of \mathbb{N} . By applying diagonalization argument to (M_t) , we get an infinite subset M of \mathbb{N} such that $M \setminus M_t$ is finite for all $t \in \mathbb{N}$. Now, it is easy to see that the restriction of T to $[x_n]_{n \in M}$ is an isomorphism and $[x_n]_{n \in M}$ is nuclear that is, (ii) holds.

□

Corollary 1. *Let E be a metrizable locally convex space with a basis $(x_n)_{n \in \mathbb{N}}$ with increasing continuous norms $\|\cdot\|_k$ for all $k \in \mathbb{N}$ and F be a countably normed Frechet space with increasing continuous seminorms $|\cdot|_t$ for all $t \in \mathbb{N}$ which gives topologies of the respective spaces. Let the identity operator I_F on F , have a decomposition $I_F = S_1 + S_2 + \cdots + S_l$ into some continuous linear maps $S_i : F \rightarrow F$ for all $1 \leq i \leq l$ for some $l \in \mathbb{N}$. Let $T : E \rightarrow F$ be a continuous linear operator from E into F . Then there exists an infinite*

subset M of \mathbb{N} such that either

$$(a) \left(\frac{Tx_n}{\|x_n\|_k} \right)_{n \in M} \text{ is a bounded subset of } F \text{ for some } k \in \mathbb{N},$$

or

$$(b) \text{ restriction of } S_i T \text{ to } [x_n]_{n \in M} \text{ is an isomorphism for some } i \in \{1, 2, \dots, l\}.$$

Proof. WLOG, we may assume $I_F = S_1 + S_2$. Apply lemma 2 to $S_1 T$ and suppose the conclusion (b) does not hold. Then, $(\frac{S_1 T x_n}{\|x_n\|_1})_{n \in M}$ is bounded for some infinite subset M of \mathbb{N} . Now apply lemma 2 to the restriction of $S_2 T$ to $[(x_n)]_{n \in M}$. If (i) in lemma 2 holds, then conclusion (a) holds. Otherwise (b) holds. \square

Proposition 2. *Let F be a Frechet space with an unconditional basis $(e_i)_{i \in \mathbb{N}}$, satisfying $e_i^*(e_j) = \delta_{ij}$ for $e_i^* \in F'$, which admits a continuous norm. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of nonzero vectors in F such that the number of elements of the set $A_n = \{i \in \mathbb{N} : e_i^*(x_n) \neq 0\}$ is at most k for each $n \in \mathbb{N}$ for some $k \in \mathbb{N}$ and $A_n \cap A_m = \emptyset$ when $n \neq m$. If F is Montel or isomorphic to some Köthe space $\lambda^p(A)$ with the canonical base mentioned above, then there is a complemented subsequence $(x_{n_m})_{m \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$.*

Proof. Let $(L_i)_{i=1}^k$ be a partition of \mathbb{N} such that $L_i \cap A_n$ has at most one element for each $n \in \mathbb{N}$ and $1 \leq i \leq k$. Set $F_i := [e_n]_{n \in L_i}$ and $E := [x_n]_{n \in \mathbb{N}}$ and $T : E \rightarrow F := \oplus_{i=1}^k F_i$ be the inclusion map. Then by Lemma 2, there exists an infinite subset M of \mathbb{N} such that either (i) $\left(\frac{x_n}{\|(x_n)\|} \right)_{n \in M}$ is bounded where $\|\cdot\|$ is a continuous norm on F or (ii) the restriction of natural projection P_j from F onto F_j is an isomorphism of $[x_n]_{n \in M}$ for some $j \in \{1, 2, \dots, k\}$.

If (ii) holds then $[x_n]_{n \in M}$ is complemented in F by Lemma 1.

Now, if F is Montel, then (i) does not hold. Therefore, $(x_n)_{n \in M}$ is a complemented subsequence of $(x_n)_{n \in \mathbb{N}}$.

If $F \simeq \lambda^p(A)$ and (i) holds then $[x_n]_{n \in \mathbb{N}}$ is isomorphic to l_p and $P(x) := \sum_{n \in M} x_n^*(x) x_n$ is a projection from F onto $[x_n]_{n \in M}$ where $x_n^*(x_m) = \delta_{nm}$ for $x_n^* \in F'$ and $\|x_n^*\| \leq r$ for each $n, m \in M$ and for some $r \in \mathbb{R}$. \square

Corollary 2. *Let $(E_n)_{n \in \mathbb{N}}$ be a two dimensional decomposition of a nuclear Frechet space E . Then each sequence of nonzero vectors $(x_{E_n})_{n \in \mathbb{N}}$ has a complemented subsequence.*

Proof. Since E is a nuclear Frechet space with 2-FDD, there exists a sequence $(y_{E_n})_{n \in M}$ for some infinite subset M of \mathbb{N} such that $[y_{E_n}]_{n \in M}$ has a complementary basis [5], which can be made naturally complemented by , say $[z_{E_n}]_{n \in M}$. Thus, we may write $E_n = [y_{E_n}] + [z_{E_n}]$ and $E_M = [y_{E_n}]_{n \in M} \oplus [z_{E_n}]_{n \in M}$.

Let P be the projection map on E with range $[y_{E_n}]_{n \in M}$. Then

$$P^{-1}(0) = [z_{E_n}]_{n \in M} \bigoplus \bigoplus_{n \in \mathbb{N} \setminus M} E_n.$$

Recall that $E_M := [E_n]_{n \in M}$ and note $I_M = P|_M + (I - P)|_M$. By applying proposition 2 to the space $[x_n]_{n \in M}$ and taking I_M to be T , we obtain the fact that x_{E_n} has a complemented subsequence. \square

Let E be a locally convex space such that $E = \overline{\bigoplus_{n \in \mathbb{N}} E_n}$ where $\dim(E_n) = k$ for each $n \in \mathbb{N}$ and for some $k \in \mathbb{N}$. We say that E satisfies *condition (*)*, if F is any 2-dimensional step quotient of step subspace of E with decomposition $F = \bigoplus_{n \in \mathbb{N}} F_n$ contains closed subspaces G_x for any $(x_{F_n})_{n \in \mathbb{N}}$ such that $F_M = [x_{F_n}]_{n \in M} + G_x$ and $\dim(G_x \cap F_n) = 1$ for some infinite subset M of \mathbb{N} and for all $n \in \mathbb{N}$.

Proposition 3. *Let E be a locally convex space such that $E = \overline{\bigoplus_{n \in \mathbb{N}} E_n}$ where $\dim(E_n) = k$ for each $n \in \mathbb{N}$, for some $k \in \mathbb{N}$, and satisfy condition (*). Then, there are closed subspaces G_1, G_2, \dots, G_k of E and an infinite subset M of \mathbb{N} . such that*

$$E_M = G_1 + G_2 + \dots + G_k \quad \text{where} \quad \dim(G_i \cap E_n) = 1 \text{ for all } n \in M, \quad 1 \leq i \leq k.$$

Proof. Let E be a locally convex space such that $E = \overline{\bigoplus_{n \in \mathbb{N}} E_n}$ where $\dim(E_n) = k$ for each $n \in \mathbb{N}$, for some $k \in \mathbb{N}$, and satisfy condition (*). We proceed with induction on k . For the case $k = 1$, it is immediate.

Let $(x_{E_n})_{n \in \mathbb{N}}$ be nonzero vectors. Let $Q : E \rightarrow E/[x_{E_n}] =: F$ be the quotient map. Thus, F satisfies condition (*) with $\dim(F_n) = k - 1$. By

induction hypothesis, we have closed subspaces G_1, G_2, \dots, G_{k-1} of F such that

$$F_M = G_1 + G_2 + \dots + G_{k-1} \quad \text{where } \dim(G_i \cap F_n) = 1 \text{ for all } n \in \mathbb{N}, 1 \leq i \leq k-1,$$

and for some infinite subset M of \mathbb{N} . Set $H_n^i := Q^{-1}(G_i) \cap E_n$ which are 2-dimensional subspaces of E containing x_{E_n} for all $n \in M$ and for $1 \leq i \leq k-1$.

Define $\mathcal{H}_i := [H_n^i]_{n \in M}$. Then \mathcal{H}_i is 2-FDD step subspace of $[E_n]_{n \in M}$, a space which satisfy condition (*) for $1 \leq i \leq k-1$. If we consider $(x_{E_n})_{n \in M}$ and \mathcal{H}_1 in conjunction with condition (*), we conclude that there exists a closed subspace \mathcal{G}_1 of E_M such that $[\mathcal{H}_1]_{M_1} = [x_{E_n}]_{n \in M_1} + \mathcal{G}_1$ for some infinite subset M_1 of M .

Now consider $(x_{E_n})_{n \in M}$ and \mathcal{H}_2 to obtain a closed subspace \mathcal{G}_2 of \mathcal{H}_1 such that $[\mathcal{H}_2]_{M_2} = [x_{E_n}]_{n \in M_2} + \mathcal{G}_2$ for some infinite subset M_2 of M_1 .

After proceeding in this manner, we end up with

$$E_L = [x_{E_n}]_{n \in L} \oplus \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \dots \oplus \mathcal{G}_{k-1},$$

where $L := M_{k-1} \subseteq M_{k-2} \subseteq \dots \subseteq M_1 \subseteq M \subseteq \mathbb{N}$.

□

Proposition 4. *Let E be a Frechet space with a k -dimensional unconditional decomposition $(E_n)_{n \in \mathbb{N}}$ and suppose E satisfies condition (*). Then any sequence $(x_{E_n})_{n \in \mathbb{N}}$ has a naturally complemented subsequence in E .*

Proof. Let E be a Frechet space with a k -dimensional unconditional decomposition $(E_n)_{n \in \mathbb{N}}$ and suppose E satisfies condition (*). Then, there are closed subspaces G_1, G_2, \dots, G_k of E and an infinite subset M of \mathbb{N} such that

$$E_M = G_1 + G_2 + \dots + G_k \quad \text{where } \dim(G_i \cap E_n) = 1 \text{ for all } n \in M, 1 \leq i \leq k,$$

and $G_1 = [x_n]_{n \in M}$ by proposition 3. Graph of projections P_i with ranges G_i , respectively, are closed, thus, by closed graph theorem each projection P_i is continuous, for $1 \leq i \leq k$. Thus, by lemma 1, E_M is complemented.

Therefore, any sequence of nonzero vectors (x_{E_n}) has a complemented subsequence.

□

The next result follows from Corollary 2 and Proposition 3.

Theorem 1. *Let E be a nuclear Frechet space with a k -dimensional decomposition $(E_n)_{n \in \mathbb{N}}$. Then any sequence of nonzero vectors $x_n \in E_n$ has a complemented subsequence and the subspace $[E_n]_{n \in M}$ of E has a basis for some infinite subset M of \mathbb{N} .*

4 Some generalizations

So far, we have dealt with nuclear Frechet spaces. Now we show that the result obtained in Theorem 1 can be generalized to Montel-Frechet spaces and then we turn our attention to Frechet spaces with unconditional FDD which does not admit a continuous norm.

Proposition 5. *Let $E = [E_n]_{n \in \mathbb{N}}$ be a Montel-Frechet space with unconditional k -FDD with continuous norm. Then any sequence (x_{E_n}) has a naturally complemented subsequence. Furthermore, there exists an infinite subset M of \mathbb{N} such that $E_M = \lambda(A)$ for some nuclear Kothe space $\lambda(A)$.*

Proof. It is sufficient to show that E_M is nuclear for some $M \subseteq \mathbb{N}$ by Theorem 1.

Let x_{E_n} be a sequence of nonzero vectors. By applying Lemma 2 to I_E , we end up the second conclusion in the lemma 2 holds, that is, $[x_{E_n}]_{L \in \mathbb{N}}$ is nuclear for some infinite subset L of \mathbb{N} , for E is Montel. We proceed by induction on k .

For $k = 1$, it is immediate.

Now define $[F_n]_{n \in L} := [E_n]_{n \in L} / [x_{E_n}]_{L \in \mathbb{N}}$. $[F_n]_{n \in L}$ is a Montel space, since $[E_n]_{n \in L}$ is Montel and $[x_{E_n}]_{L \in \mathbb{N}}$ is nuclear. Also, note that $[F_n]_{n \in L}$ has $k - 1$ -FDD and is with continuous norm. By induction hypothesis, there exists an infinite subset M of \mathbb{N} such that F_M is nuclear.

Therefore, E_M is nuclear by the three space property of nuclearity. \square

Now suppose E is not Montel but has an absolute p -basis instead of an unconditional one with k -FDD and continuous norm. Then, one can show that any sequence (x_{E_n}) has a naturally complemented subsequence in that space. In fact, there is M , such that either $E_M = \lambda(A)$ or $E_M = l_p$ or $E_M = \lambda(A) \times l_p$ for $1 \leq p < \infty$ or $p = 0$, that is, c_0 in view of Lemma 2.

If E is a subspace of $l_p^{\mathbb{N}}$ or $c_0^{\mathbb{N}}$ with strong FDD, using a modification of Bessaga-Pelczynski selection principle for the finite rank operators in Frechet

spaces, we may suppose E to have an absolute p-basis and proceed as in the above paragraph.

As for spaces which do not admit continuous norms, we have the following which can be considered as a similar result for spaces with FDD instead of spaces with unconditional basis which was obtained by Floret and Moscatelli in [7].

Proposition 6. *Let E be a Frechet space with FDD and with a sequence of projections $\{P_n : n \in \mathbb{N}\}$ and a fundamental system of seminorms $\|\cdot\|_k$. WLOG, we may assume $\|P_n(x)\|_k \leq \|x\|_k$ for all $k \in \mathbb{N}$ and for all $x \in E$. Then, either E has a continuous norm or $E = \oplus E_{M_i}$ unconditionally where each E_{M_i} admits continuous norms for $M_i \subseteq \mathbb{N}$ for all $i \in \mathbb{N}$.*

Proof. Let $E_n := P_n(E)$ for all $n \in \mathbb{N}$. Define

$$\varphi(n) := \min\{k : \|\cdot\|_k \text{ is a norm on } E_n\}.$$

Then, set $M_k := \{n \in \mathbb{N} : \varphi(n) = k\}$. Now, either $\mathbb{N} = \bigcup_{k=1}^L M_k$ for some $L \in \mathbb{N}$ or $\mathbb{N} = \bigcup_{k=1}^{\infty} M_k$. In the former case, E admits a continuous norm. In the latter case, $E = \oplus E_{M_k}$ with $\|\cdot\|_k$ is a norm on E_{M_k} . Note that if each M_k is finite then $E = \omega$.

□

If $E \neq \omega$ but have a FDD with unconditional basis, then there exists an infinite subset M of \mathbb{N} such that E_M admits a continuous norm. Furthermore, if E has a strong FDD, the above results are applicable for E_M .

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